SOPHUS LIE.*

BY PROFESSOR GASTON DARBOUX.

Sophus Lie was born on the 17th of December, 1842, at Nordfjordeid (near Florö) where his father, John Herman Lie, was pastor. The studies of his childhood and youth did not reveal in him that exceptional aptitude for mathematics which is signalized so early in the lives of the great geometers: Gauss, Abel, and many others. Even on leaving the University of Christiania in 1865, he still hesitated between philology and mathematics. It was the works of Plücker on modern geometry which first made him fully conscious of his mathematical abilities and awakened within him an ardent desire to consecrate himself to mathematical research. Surmounting all difficulties and working with indomitable energy he published his first work in 1869, and we can say that from 1870 on he was in possession of the ideas which were to direct his whole career.

At this time I frequently had the pleasure of meeting and conversing with him in Paris where he had come with his friend F. Klein. A course of lectures by Sylow revealed to Lie all the importance of the theory of substitution groups; the two friends studied this theory in the great treatise of our colleague Jordan; they saw fully the essential rôle which it would be called upon to play in all the branches of mathematics to which it had then not been applied. They have both had the good fortune to contribute by their works to impressing upon mathematical studies the direction which appeared to them to be the best.

A short note of Lie "Sur une transformation géométrique," presented to our Academy in October, 1870, contains an extremely original discovery. Nothing resembles a sphere less than a straight line and yet, by using the ideas of Plücker, Lie found a singular transformation which makes a sphere correspond to a straight line, and which consequently makes possible the derivation of a theorem relative to an ensemble of spheres from every theorem relative to an aggregate of straight lines, and vice versa. It is true that if the lines are real, the corresponding spheres are imaginary. But such difficulties are not sufficient to deter geometers. In this curious method of transformation, each property relative to asymptotic lines of a surface is transformed into a property relative to lines of curvature. The name of Lie will remain attached to these concealed relations which connect the two essential and fundamental elements of geometric investigation, the straight line and sphere. He has developed them in detail in a memoir full of new ideas which appeared in 1872 in the *Mathematische Annalen*.

The works following this brilliant beginning fully confirmed all the hopes to which it gave birth. Since the year 1872 Lie has put forth a series of memoirs upon the most difficult and most advanced parts of the integral calculus. He

*From the Bulletin of the American Mathematical Society. Translated by Edgar Odell Lovett from *Comptes rendus*.

commences by a profound study of the works of Jacobi on the partial differential equations of the first order and at first coöperates with Mayer in perfecting this theory in an essential point. Then, by continuing the study of this beautiful subject, he is led to construct progressively that masterful theory of continuous transformation groups which constitutes his most important work and in which, at least at the start, he was aided by no one. The detailed analysis of this vast theory would require too much space here. It is proper, however, to point out particularly two elements wholly essential to these researches: first, the use of contact transformations which throws such a vivid and unexpected light upon the most difficult and obsure parts of the theories relative to the integration of partial differential equations; second, the use of infinitesimal transformations. The introduction of these transformations is due entirely to Lie; their use, like that of Lagrange's variation, naturally greatly extends both the notion of differential and the applications of the infinitesimal calculus.

The construction of so extended a theory did not satisfy Lie's activity. In order to show its importance he has applied it to a great number of particular subjects, and each time he has had the good fortune of meeting with new and elegant properties. I find my preference in the researches which he has published since 1876 on minimal surfaces. The theory of these surfaces, the most attractive perhaps that presents itself in geometry, still awaits, and may await a long time, the complete solution of the first problem to be proposed in it, namely, the determination of a minimal surface passing through a given contour. But, in return, it has been enriched by a great number of interesting propositions due to a multitude of geometers. In 1866 Weierstrass made known a very precise and simple system of formulæ which has called forth a whole series of new studies on these surfaces. In his works Lie returns simply to the formulæ of Monge; he gives their geometric interpretation and shows how their use can lead to the most satisfactory theory of minimal surfaces. He makes known methods which permit of determining all algebraic minimal surfaces of given class and order. Finally, he studies the following problem: to determine all algebraic minimal surfaces inscribed in a given algebraical developable surface. He gives the complete solution for the case where only one of these surfaces inscribed in the developable is known.

Of great interest also are the researches which we owe to him on the surfaces of constant curvature, in the study of which he makes use of a theorem of Bianchi on geodesic lines and circles, likewise those on surfaces of translation, on the surfaces of Weingarten, on the equations of the second order having two independent variables, et cetera. I should reproach myself for forgetting, even in so rapid a résumé, the applications which Lie has made of his theory of groups to the non-Euclidean geometry and to the profound study of the axioms which lie at the basis of our geometric knowledge.

These extensive works quickly attracted to the great geometer the attention of all those who cultivate science or are interested in its progress. In 1877 a new chair of mathematics was created for him at the University of Christiania,

and the foundation of a Norwegian review enabled him to pursue his work and publish it in full. In 1886, he accepted the honor of a call to the University of Leipzig; he taught in this university with the rank of ordinary professor from 1886 to 1898. To this period of his life is to be referred the publication of his didactic works, in which he has coördinated all his researches. Six months ago he returned to his native land to assume at Christiania the chair which had been especially reserved for him by the Norwegian parliament, with the exceptional salary of ten thousand crowns. Unfortunately, excess of work had exhausted his strength and he died of cerebal anæmia at the age of fifty-six years.

Nowhere is his loss felt more keenly than in our country, where he had so many friends. True, in 1870 a misadventure befell him, whose consequences I was instrumental in averting. Surprised at Paris by the declaration of war, he took refuge at Fontainebleau. Occupied incessantly by the ideas fermenting in his brain, he would go every day into the forest, loitering in places most remote from the beaten path, taking notes and drawing figures. It took little at this time to awaken suspicion. Arrested and imprisioned at Fontainebleau, under conditions otherwise very comfortable, he called for the aid of Chasles, Bertrand, and others; I made the trip to Fontainebleau and had no trouble in convincing the procureur impérial; all the notes which had been seized and in which figured complexes, orthogonal systems, and names of geometers, bore in no way upon the national defenses. Lie was released; his high and generous spirit bore no grudge against our country. Not only did he return voluntarily to visit it but he received with great kindness French students, scholars of our École Normale who would go to Leipzig to follow his lectures. It is to the École Normale that he dedicated his great work on the theory of transformation groups. A number of our thesis at the Sorbonne have been inspired by his teaching and dedicated to him.

The admirable works of Sophus Lie enjoy the distinction, to-day quite rare, of commanding the common admiration of geometers as well as analysts. He has discovered fundamental propositions which will preserve his name from oblivion, he has created methods and theories which, for a long time to come, will exercise their fruitful influence on the development of mathematics. The land where he was born and which has known how to honor him can place with pride the name of Lie beside that of Abel, of whom he was a worthy rival and whose approaching centenary he would have been so happy in celebrating.

ON THE SIMPLE GROUPS WHICH CAN BE REPRESENTED AS SUBSTITUTION GROUPS THAT CONTAIN CYCLICAL SUBSTITUTIONS OF A PRIME DEGREE.

By DR. G. A. MILLER.

It is known that cyclical substitutions of a prime degree (p) cannot occur in any primitive group (if it is not alternating or symmetric) unless the degree of the primitive group (G) is one of the following three numbers, p, $p+1$, $p+2$. When G is of degree p all its substitutions of order p generate a simple group which is a selfconjugate subgroup of G (if it does not coincide with G) and corresponds to identity of a cyclical quotient group of G, the order of this quotient group being a divisor of $p-1$.* We proceed to consider the cases when G contains substitutions of order p and is either of degree $p+1$ or of degree $p+2$.

If G is of degree $p+2$ and is not generated by its substitutions of order p it must contain a selfconjugate subgroup (G') which is generated by these substitutions, G' is at least triply transitive since it contains substitutions of order p.† We proceed to prove that it is a simple group when p exceeds 2. If it were compound each of its at least doubly transitive maximal subgroups of degree $p+1$ would also be compound, since the substitutions of such a subgroup (G_1) which would belong to the given selfconjugate subgroup of G, would form a selfconjugate subgroup of G_1.

A maximal subgroup (G_2) of degree p that is contained in G_1 must be transitive. Since every selfconjugate subgroup of a transitive group of degree p must contain substitutions of order p, G_2 could have only identity in common with the given self-conjugate subgroup of G. Hence the order of this selfconjugate subgroup would be $(p+1)(p+2)$, and the corresponding quotient group would be simply isomorphic to G_2. To a subgroup of order p that is contained in G_2 there would correspond a subgroup of G, whose order would be $p(p+1)(p+2)$. Since this is evidently impossible we have the important

THEOREM. *All the substitutions of order p (p being any odd prime number) that are contained in any primitive group of degree p or of degree $p+2$ generate a simple group. If this simple group does not coincide with the entire group it is self-conjugate and the corresponding quotient group is cyclical and has for its order a divisor of $p-1$.‡ This simple group cannot be selfconjugate subgroup of more than one group of its own degree and of a given order.*

We shall now consider the groups of degree $p+1$, p being any odd prime number, that contain substitutions of order p. Such groups are at least doubly transitive and their substitutions of order p generate a doubly transitive selfconjugate subgroup if they do not generate the entire group. Let H be such a sub-

*Cf. *Bulletin of the American Mathematical Society*, Vol. 4, 1898, page 140.

†Jordan, *Journal de Mathematiques*, Vol. 16, 1871.

‡From this theorem it follows directly that the primitive group of degree 9 and of order 504 is simple.

group. We see as in the preceding paragraph that H cannot contain any self-conjugate subgroup unless this subgroup is a regular group of order $p+1$. As this regular subgroup has to contain p subgroups that are conjugate in H it cannot involve any substitution whose order exceeds 2. Hence we have the

THEOREM. *If $p+1$ is not a power of 2 then the substitutions of order p (p being any odd prime number) that are contained in a group of degree $p+1$ generate a simple group. If this simple group does not coincide with the entire group it is selfconjugate and the corresponding quotient group is a cyclical and its order is a divisor of $p-1$.* If $p+1$ is a power of 2, the group generated by the substitutions of order p that are contained in a group of degree $p+1$ cannot contain any selfconjugate subgroup except perhaps the regular group of order $p+1$ which contains no substitution whose order exceeds 2.*

Cornell University, April 3, 1899.

*From this theorem it follows directly that the three primitive groups of degree 12 and orders 660, 7920, and 95040, respectively, are simple. The last of these three groups is the well known five-fold transitive group of Mathien.

ON SYMMETRIC FUNCTIONS.

By E. D. ROE, Jr., Associate Professor of Mathematics in Oberlin College.

[Continued from March Number.]

B. FUNDAMENTAL RELATIONS FOR SYMMETRIC FUNCTIONS.

1. FUNDAMENTAL RELATIONS BETWEEN COEFFICIENTS.

(1). *Derivation of the relations.*

In A, 4, (3) we have already obtained one of the relations, viz:

$$\binom{0^{\lambda_0}1^{\lambda_1}\ldots\ldots\ldots n^{\lambda_n}}{b_0{}^m\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ldots\beta_m{}^{\kappa_n}} = (-1)^{mn}\binom{(m-\kappa_1)(m-\kappa_2)\ldots\ldots(m-\kappa_n)}{a_0{}^n(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots\ldots(\alpha\lambda_n)^0}.$$

If in the equations $b_0x^n+b_1x^{n-1}+\ldots\ldots+b_n'=0$, and $a_0x^m+a_1x^{m-1}+\ldots\ldots+a_m=0$ (cf. loc. cit.), we substitute $x=1/y$, b_r becomes b_{n-r}, a_r becomes a_{m-r}, and $b_0{}^m\Sigma\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ldots\beta_n{}^{\kappa_n}$ becomes

$$b_n{}^m\Sigma\frac{1}{\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ldots\beta_n{}^{\kappa_n}} = \frac{b_n{}^m}{(\beta_1\beta_2\ldots\ldots\beta_n)^m}\Sigma\beta_1{}^{m-\kappa_1}\beta_2{}^{m-\kappa_2}\ldots\ldots\beta_n{}^{m-\kappa_n}$$

$$=(-1)^{mn}b_0{}^m\Sigma\beta_1{}^{m-\kappa_1}\beta_2{}^{m-\kappa_2}\ldots\ldots\beta_n{}^{m-\kappa_n}.$$

Similarly $a_0{}^n\Sigma(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots\ldots(\alpha\lambda_n)^0$ becomes $(-1)^{mn}a_0{}^n\Sigma(\alpha\lambda_n)^n(\alpha\lambda_{n-1})^{n-1}\ldots\ldots(\alpha\lambda_0)^0$. We have therefore,

$$\begin{pmatrix} 0^{\lambda_0}1^{\lambda_1}\ldots\ldots\ldots n^{\lambda_n} \\ b_0{}^m\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ldots\beta_n{}^{\kappa_n} \end{pmatrix} = (-1)^{mn}\begin{pmatrix} 0^{\lambda_n}1^{\lambda_{n-1}}\ldots\ldots\ldots\ldots n^{\lambda_0} \\ b_0{}^m\beta_1{}^{m-\kappa_1}\beta_2{}^{m-\kappa_2}\ldots\ldots\beta_n{}^{m-\kappa_n} \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} (m-\varkappa_1)(m-\varkappa_2)\ldots\ldots(m-\varkappa_n) \\ a_0{}^n(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots\ldots(\alpha\lambda_n)^0 \end{pmatrix} = (-1)^{mn}\begin{pmatrix} \varkappa_1\varkappa_2\ldots\ldots\ldots\ldots\ldots\ldots\varkappa_n \\ a_0{}^n(\alpha\lambda_n)^n(\alpha\lambda_{n-1})^{n-1}\ldots\ldots(\alpha\lambda_0)^0 \end{pmatrix}$$

The last relations correspond to reciprocal terms in the resultant theory. [Terms of the resultant like $(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots\ldots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots\ldots(b_{s_\nu})^{q_\nu}$ and $(a_{m-r_\mu})^{p_\mu}(a_{m-r_{\mu-1}})^{p_{\mu-1}}\ldots\ldots(a_{m-r_1})^{p_1}(b_{n-s_\nu})^{q_\nu}(b_{n-s_{\nu-1}})^{q_{\nu-1}}\ldots\ldots(b_{n-s_1})^{q_1}$ are called reciprocal terms.] Since one member in each is equal numerically to the terms of the relation already obtained, the four terms are numerically equal. Taken together we shall call them the fundamental relations for the coefficients of symmetric functions.

(2). *Other notation for the fundamental relation.*

We will now re-write the relations given in (1), replacing such expressions as $b_0{}^m\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ldots\beta_n{}^{\kappa_n}$ and $a_0{}^n(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots\ldots(\alpha\lambda_n)^0$ by $0^m\varkappa_1\varkappa_2\ldots\ldots\varkappa_n$ and $0^n n^{\lambda_0}(n-1)^{\lambda_1}\ldots\ldots0^{\lambda_n}$, respectively, $(n-r)^{\lambda_r}$, *e. g.*, signifying, as before $(a\lambda_r)^{n-r}$ signified, that λ_r roots have the exponent $(n-r)$. We may then write the fundamental relations:

$$\begin{pmatrix} 0^{\lambda_0}1^{\lambda_1}\ldots\ldots n^{\lambda_n} \\ 0^m\varkappa_1\varkappa_2\ldots\ldots\varkappa_m \end{pmatrix} = \begin{pmatrix} \varkappa_1\varkappa_2\ldots\ldots\varkappa_n \\ 0^n0^{\lambda_0}1^{\lambda_1}\ldots\ldots n^{\lambda_n} \end{pmatrix} = (-1)^{mn}\begin{pmatrix} 0^{\lambda_n}1^{\lambda_{n-1}}\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^m(m-\varkappa_1)(m-\varkappa_2)\ldots(m-\varkappa_n) \end{pmatrix}$$

$$= (-1)^{mn}\begin{pmatrix} (m-\varkappa_1)(m-\varkappa_2)\ldots(m-\varkappa_n) \\ 0^n0^{\lambda_n}1^{\lambda_{n-1}}\ldots\ldots\ldots\ldots n^{\lambda_0} \end{pmatrix}^*.$$

2. Fundamental and Necessary Conditions of Coefficients.

We will next consider some relations of conditions of the coefficients themselves.

(1). *First condition.*

Since $b_0{}^m\Sigma\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ldots\beta_n{}^{\kappa_n} = b_0{}^m(\beta_1\beta_2\ldots\ldots\beta_n)^{\kappa_n}\Sigma\beta_1{}^{\kappa_1-\kappa_n}\beta_2{}^{\kappa_2-\kappa_n}\ldots\ldots(\beta_{n-1})^{\kappa_{n-1}-\kappa_n} = (-1)^{n\kappa_n}b_0{}^{m-\kappa_n}b_n{}^{\kappa_n}\Sigma\beta_1{}^{\kappa_1-\kappa_n}\beta_2{}^{\kappa_2-\kappa_n}\ldots(\beta_{n-1})^{\kappa_{n-1}-\kappa_n}$, supposing $\varkappa_1, \varkappa_2, \ldots\ldots\varkappa_n$ to be in order of descending magnitude, it follows that, at least, either $\lambda_n > 0$, (and $\lambda_n \geq \varkappa_n$), or else $\lambda_n = 0$, and $\varkappa_n = 0$. This corresponds to the condition in the resultant $R_{m,n}$, that, at least, one of the two factors b_n or a_m must be present in every term.

(2). *Second condition.*

Again, since by 1, (2),

*It might seem as if the distinction of a and b is lost by this notation. A little reflection shows that this distinction does not need to be kept. In fact the theory is clearer without it.

$$\begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} \ldots\ldots n^{\kappa_n} \\ 0^m \kappa_1 \kappa_2 \ \ldots\ldots \kappa_n \end{pmatrix} = (-1)^{mn} \begin{pmatrix} 0^{\lambda_n} 1^{\lambda_{n-1}} \ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^m (m-\kappa_1)(m-\kappa_2) \ \ldots\ldots (m-\kappa_n) \end{pmatrix}$$

it follows by the previous proof, that, at least, either $\lambda_0 > 0$, (and $\lambda_0 \geqq m - \kappa_1$), or else $\lambda_0 = 0$, $m - \kappa_1 = 0$, *i. e.*, $\kappa_1 = m$. This corresponds to the condition in the resultant $R_{m,n}$, that, at least of the two factors b_0 and a_0, one must be present in every term.

When these conditions are not satisfied, the coefficient is equal to zero. It will be shown later that if $\lambda_n = \lambda_{n-1} = \ldots = \lambda_r = 0$. then $\kappa_n = \kappa_{n-1} = \ldots = 0$, and if $\lambda_0 = \lambda_1 = \ldots = \lambda_r = 0$, then $m - \kappa_1 = m - \kappa_2 = \ldots = m - \kappa_r = 0$, *i. e.*, $\kappa_1 = \kappa_r = \ldots = \kappa_r = m$; otherwise the coefficient is zero (D, 3 and 4).

3. Fundamental Equations of Condition Connecting the Subscripts and Exponents.

From the theorems of order and weight of symmetric functions, and from the relations between the four coefficients given in 1, (2), we have the following equations of condition connecting the subscripts and exponents :

(1). $\lambda_0 + \lambda_1 = \ldots . \lambda_n = m$,

(2). $\lambda_1 + 2\lambda_2 + \ldots . n\lambda_n = \kappa_1 + \kappa_2 + \ldots . \kappa_n$,

(3). $\lambda_{n-1} + 2\lambda_{n-2} + \ldots . n\lambda_0 = mn - (\kappa_1 + \kappa_2 + \ldots . \kappa_n)$, and by adding mn to both sides of (2) that equation may be written,

(4). $\lambda_1 + 2\lambda_2 + \ldots . n\lambda_n + mn - (\kappa_1 + \kappa_2 + \ldots . \kappa_n) = mn$, the same equations which we should have in the theory of the resultant, excepting (3) which can be derived from (1) by multiplying that equation by n, and then substituting the value of $\kappa_1 + \kappa_2 + \ldots \kappa_n$ from (2).

C. NORMAL FORMS AND REDUCIBLE FORMS.

As in the theory of the resultant so also here we have normal forms and reducible forms. A normal form will be characterized by having $\lambda_0 > 0$, $\lambda_n > 0$, $\kappa_1 = m$, $\kappa_n = 0$. All other forms, the completely reducible forms excepted, may be reduced to such as are normal forms having a lower m or n. As in the rerultant theory, so here there are four kinds of reduction, with the four corresponding kinds of derivation. In the next divisions we will briefly treat of these.

D. REDUCTION.

1. Reduction in the Case where $\kappa_1 = m$.

In this case λ_0 must be greater than zero [cf. B, 2, (2)]. It is evident that we may divide by the factor b_0 in the given term as well as in $b_0^m \Sigma \beta_1^{\kappa_1} \beta_2^{\kappa_2} \ldots . \beta_n^{\kappa_n}$ without affecting the value of the coefficient of the term, and that therefore,

$$\begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} \ \ldots . \ n^{\lambda_n} \\ 0^m \kappa_1 \kappa_2 \ \ldots\ldots \kappa_n \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0 - 1} 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^{m-1} \kappa_1 \kappa_2 \ \ldots\ldots \kappa_n \end{pmatrix},$$

and this reduction may be continued until the exponent of b_0 below is equal to $\varkappa_1$. We then have

$$\begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m \varkappa_1 \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0 - \lambda} 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^{\varkappa_1} \varkappa_1 \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix},$$

where $m - \lambda = \varkappa$, $\lambda_0 - \lambda \geqq 0$.

This reduction corresponds to the third kind of reduction in the theory of the resultant.

2. Reduction in the Case where $\varkappa_n \neq 0$.

Here we must have $\lambda_n > 0$. [Cf. B, 2, (1)], and the work already done in B, 2, shows that

$$\begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m \varkappa_1 \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = (-1)^{n\varkappa_n} \begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} \ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_n - \varkappa_n} \\ 0^{m-\varkappa_n} (\varkappa_1 - \varkappa_n)(\varkappa_2 - \varkappa_n) \ldots (\varkappa_{n-1} - \varkappa_n) \end{pmatrix}$$

or we may prove it otherwise as follows: By B, 1, (2),

$$= \begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m \varkappa_1 \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = (-1)^{m\varkappa} \begin{pmatrix} 0^{\lambda_n} 1^{\lambda_{n-1}} \ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^m (m - \varkappa_1)(m - \varkappa_2) \ldots\ldots (m - \varkappa_n) \end{pmatrix}.$$

By 1, the right member of this can be reduced. We get

$$\begin{pmatrix} 0^{\lambda_n} 1^{\lambda_{n-1}} \ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^m (m - \varkappa_n)(m - \varkappa_{n-1}) \ldots\ldots (m - \varkappa_1) \end{pmatrix} = \begin{pmatrix} 0^{\lambda_n - \varkappa_n} 1^{\lambda_{n-1}} \ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^{m-\varkappa_n} (m - \varkappa_n)(m - \varkappa_{n-1}) \ldots\ldots (m - \varkappa_1) \end{pmatrix}.$$

Again by B, 1, (2), $\begin{pmatrix} 0^{\lambda_n - \varkappa_n} 1^{\lambda_{n-1}} \ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^{m-\varkappa_n} (m - \varkappa_n)(m - \varkappa_{n-1}) \ldots (m - \varkappa_1) \end{pmatrix}$

$$= (-1)^{(m-\varkappa_n)n} \begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_n - \varkappa_n} \\ 0^{m-\varkappa_n} (\varkappa_1 - \varkappa_n)(\varkappa_2 - \varkappa_n) \ldots (\varkappa_{n-1} - \varkappa_n) \end{pmatrix}.$$

Substituting, we get

$$\begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m \varkappa_1 \varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = (-1)^{n\varkappa_n} \begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_n - \varkappa_n} \\ 0^{m-\varkappa_n} (\varkappa_1 - \varkappa_n)(\varkappa_2 - \varkappa_n) \ldots\ldots (\varkappa_{n-1} - \varkappa_n) \end{pmatrix}$$

as before. This corresponds to the first kind of reduction in the resultant theory.

3. Reduction in the Case where $\lambda_n = 0$.

We must have $\varkappa_n = 0$ in this case. By B, 1, (2) we have

$$\begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m \varkappa_1 \varkappa_2 \ldots\ldots \varkappa_{n-1} 0 \end{pmatrix} = \begin{pmatrix} 0 & \varkappa_{n-1} & \varkappa_{n-2} \ldots & \varkappa_1 \\ 0^n & 0^{\lambda_0} & 1^{\lambda_1} \ldots\ldots & n^0 \end{pmatrix}.$$

By D, 1, the right member can be reduced, and

$$\begin{pmatrix} 0 & \varkappa_{n-1} & \varkappa_{n-2} \ldots\ldots & \varkappa_1 \\ 0^n & 0^{\lambda_0} & 1^{\lambda_1} \ldots (n-1)^{\lambda_{n-1}} & n^0 \end{pmatrix} = \begin{pmatrix} 0 & \varkappa_{n-1} & \varkappa_{n-2} \ldots\ldots\ldots & \varkappa_1 \\ 0^n & 0^{\lambda_0} & 1^{\lambda_1} \ldots\ldots & (n-1)^{\lambda_{n-1}} \end{pmatrix}$$

$$=\begin{pmatrix}\varkappa_{n-1}\,\varkappa_{n-2}\ldots\ldots\ldots\ldots\varkappa_1\\ 0^{n-1}\,0^{\lambda_0}\,1^{\lambda_1}\ldots(n-1)^{\lambda_{n-1}}\end{pmatrix}.$$

Again by B, 1, (2),

$$\begin{pmatrix}\varkappa_{n-1}\,\varkappa_{n-2}\ldots\ldots\ldots\varkappa_1\\ 0^{n-1}0^{\lambda_0}\,1^{\lambda_1}\ldots(n-1)^{\lambda_{n-1}}\end{pmatrix}=\begin{pmatrix}0^{\lambda_0}\,1^{\lambda_1}\ldots(n-1)^{\lambda_{n-1}}\\ 0^m\varkappa_1\,\varkappa_2\ldots\ldots\varkappa_{n-1}\end{pmatrix},$$

and by substituting,

$$\begin{pmatrix}0^{\lambda_0}\,1^{\lambda_1}\ldots(n-1)^{\lambda_{n-1}}n^0\\ 0^m\varkappa_1\,\varkappa_2\ldots\ldots\varkappa_{n-1}\,0\end{pmatrix}=\begin{pmatrix}0^{\lambda_0}\,1^{\lambda_1}\ldots(n-1)^{\lambda_{n-1}}\\ 0^m\varkappa_1\,\varkappa_2\ldots\ldots\varkappa_{n-1}\end{pmatrix}.$$

In a similar way it will follow that if $\lambda_n=\lambda_{n-1}=\ldots=\lambda_r=0$, $\varkappa_n=\varkappa_{n-1}=\ldots=\varkappa_r=0$, and

$$\begin{pmatrix}0^{\lambda_0}\,1^{\lambda_1}\ldots r^0(r+1)^0\ldots n^0\\ 0^m\varkappa_1\,\varkappa_2\ldots\ldots\varkappa_{r-1}0^{n-r+1}\end{pmatrix}=\begin{pmatrix}0^{\lambda_0}\,1^{\lambda_1}\ldots(r-1)^{\lambda_{r-1}}\\ 0^m\varkappa_1\,\varkappa_2\ldots\ldots\varkappa_{r-1}\end{pmatrix}.$$

The meaning of this last formula is that a function of the roots of an equation of the nth degree, which involves only $(r-1)$ roots at a time, has the same coefficient of the term involving $b_0^{\lambda_0}b_1^{\lambda_1}\ldots(b_{r-1})^{\lambda_{r-1}}$, as the same function of the roots of an equation of the $(r-1)$st degree which involves all the roots; and the reduction corresponds to the reduction of the second kind in the resultant theory.

4. Reduction in the Case where $\lambda_0=0$.

We must have here $\varkappa_1=m$. By B, 1, (2), and D, 3,

$$\begin{pmatrix}1^{\lambda_1}2^{\lambda_2}\ldots\ldots n^{\lambda_n}\\ 0^{\varkappa_1}\,\varkappa_1\,\varkappa_2\ldots\ldots\varkappa_n\end{pmatrix}=(-1)^{\varkappa_1,n}\begin{pmatrix}0^{\lambda_n}1^{\lambda_{n-1}}\ldots\ldots\ldots\ldots(n-1)^{\lambda_1}\\ 0^{\varkappa_1}(\varkappa_1-\varkappa_n)(\varkappa_1-\varkappa_{n-1})\ldots\ldots(\varkappa_1-\varkappa_2)\end{pmatrix}.$$

Again by B, 1, (2),

$$\begin{pmatrix}0^{\lambda_n}\,1^{\lambda_{n-1}}\ldots\ldots\ldots\ldots\ldots(n-1)^{\lambda_1}\\ 0^{\varkappa_1}\,(\varkappa_1-\varkappa_n)(\varkappa_1-\varkappa_{n-1})\ldots\ldots(\varkappa_1-\varkappa_2)\end{pmatrix}=(-1)^{\varkappa_1(n-1)}\begin{pmatrix}0^{\lambda_1}\,1^{\lambda_2}\ldots(n-1)^{\lambda_n}\\ 0^{\varkappa_1}\,\varkappa_2\,\varkappa_3\ldots\ldots\varkappa_n\end{pmatrix}.$$

Therefore $\begin{pmatrix}1^{\lambda_1}\,2^{\lambda_2}\ldots\ldots n^{\lambda_n}\\ 0^{\varkappa_1}\,\varkappa_1\,\varkappa_2\ldots\ldots\varkappa_n\end{pmatrix}=(-1)^{\varkappa_1}\begin{pmatrix}0^{\lambda_1}\,1^{\lambda_2}\ldots\ldots(n-1)^{\lambda_n}\\ 0^{\varkappa_1}\,\varkappa_2\,\varkappa_3\ldots\ldots\ldots\varkappa_n\end{pmatrix}$.

If $\lambda_0=\lambda_1=\ldots=\lambda_{r-1}=0$.

$$\begin{pmatrix}r^{\lambda_r}(r+1)^{\lambda_{r+1}}\ldots n^{\lambda_n}\\ 0^m\,\varkappa_1\,\varkappa_2\ldots\ldots\ldots\varkappa_n\end{pmatrix}=(-1)^{mn}\begin{pmatrix}0^{\lambda_n}\,1^{\lambda_{n-1}}\ldots\ldots\ldots\ldots(n-r)^{\lambda_r}\\ 0^m(m-\varkappa_1)(m-\varkappa_2)\ldots\ldots(m-\varkappa_n)\end{pmatrix},$$

and the reduction of this section gives

$$\begin{pmatrix}r^{\lambda_r}(r+1)^{\lambda_{r+1}}\ldots n^{\lambda_n}\\ 0^m\varkappa_1\,\varkappa_2\ldots\ldots\ldots\varkappa_n\end{pmatrix}=(-1)^{\varkappa_1 r}\begin{pmatrix}0^{\lambda_r}\,1^{\lambda_{r+1}}\ldots\,(n-r)^{\lambda_n}\\ 0^{\varkappa_1}\,\varkappa_{r+1}\,\varkappa_{r+2}\ldots\ldots\varkappa_n\end{pmatrix}.$$

This kind of reduction corresponds to reduction of the fourth kind in the resultant theory.

[To be continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

107. Proposed by REBER V. ALLEN, Hooker Station, Ohio.

A barn, $ABCD$, length $AB=b$ feet, width $AD=a$ feet, standing in an open field, has a horse tethered to a point, P, in the side, AB, distance $AP=c$ feet, with a rope R feet long. Over what area can the horse graze?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

We might consider six cases of which only the fifth and sixth are represented in the figure.

Case I. In every case $c<b-c$. $R>c$ and $<b-c$, also $<a+c$.

Area $=\frac{1}{2}\pi R^2+\frac{1}{4}\pi(R-c)^2=\frac{1}{4}\pi(3R^2-2Rc+c^2)$.

Case II. $R>c$, $R>b-c$, $R<a+c$, $R<a+b-c$.

Area $=\frac{1}{2}\pi R^2+\frac{1}{4}\pi(R-c)^2+\frac{1}{4}\pi(R-b+c)^2=\frac{1}{4}\pi(4R^2+2c^2+b^2-2Rb-2bc)$.

Case III. $R>c+a$, $R<a+b-c$.

Area $=\frac{1}{2}\pi R^2+\frac{1}{4}\pi(R-c)^2+\frac{1}{4}\pi(R-c-a)^2+\frac{1}{4}\pi(R-b+c)^2=\frac{1}{4}\pi(5R^2-2Rc+3c^2+a^2-2Ra+2ac+b^2-2Rb-2bc)$.

Case IV. $R>c+a$, $R>a+b-c$, $R<a+b$.

Area $=\frac{1}{2}\pi R^2+\frac{1}{4}\pi(R-c)^2+\frac{1}{4}\pi(R-c-a)^2+\frac{1}{4}\pi(R-b+c)^2+\frac{1}{4}\pi(R-a-b+c)^2=\frac{1}{2}\pi(3R^2+2c^2+a^2-Ra+b^2-2Rb-2bc+ab)$.

Case V. $R>a+b$; intersection at L between AD, BC produced.

Area $=\frac{1}{2}\pi R^2+\frac{1}{4}\pi(R-c)^2+\frac{1}{4}\pi(R-b+c)^2+$sector $DTL+$sector $CIL+$triangle DLC.

$DL=R-a-c$, $DC=b$, $CL=(R-a-b+c)$.

$$\therefore\ \angle LDC=\theta=\cos^{-1}\left(\frac{b^2+(R-a-c)^2-(R-a-b+c)^2}{2b(R-a-c)}\right)$$

$$\angle DCL=\varphi=\cos^{-1}\left(\frac{b^2+(R-a-b+c)^2-(R-a-c)^2}{2b(R-a-b+c)}\right).$$

$$\therefore\ \text{Area}=\tfrac{1}{4}\pi(4R^2+2c^2+b^2-2Rb-2bc)+\tfrac{1}{2}(R-a-c)^2(\tfrac{1}{2}\pi-\theta)+\tfrac{1}{2}(R-a-b+c)^2(\tfrac{1}{2}\pi-\varphi)+\tfrac{1}{2}b(R-a-c)\sin\theta.$$

Case VI. $R>a+b$; intersection in angle KBM.

Area$=\frac{1}{4}\pi R^2+\frac{1}{4}\pi(R-c)^2+$sector $NDG+$sector $FBG+$triangle $CDG+$triangle CBG.

$DG=(R-a-c)$, $DC=b$, $BC=a$, $BG=(R-b+c)$, $DB=\sqrt{(a^2+b^2)}$.

$$\therefore\ \angle DBG=\beta=\cos^{-1}\left(\frac{a^2+b^2+(R-b+c)^2-(R-a-c)^2}{2(R-b+c)\sqrt{(a^2+b^2)}}\right)$$

$$\angle GDB=\gamma=\cos^{-1}\left(\frac{a^2+b^2+(R-a-c)^2-(R-b+c)^2}{2\sqrt{(a^2+b^2)}(R-a-c)}\right)$$

$$\angle CBD=\beta'=\cos^{-1}\left(\frac{a}{\sqrt{(a^2+b^2)}}\right)=\tan^{-1}\left(\frac{b}{a}\right)$$

$\angle CDB=\frac{1}{2}\pi-\beta'$, $\angle GDC=\gamma+\beta'-\frac{1}{2}\pi$, $\angle CBG=\beta-\beta'$, triangle DCG +triangle $BCG=$triangle $DGB-$triangle DCB.

$\therefore$ Triangle $DCG+$triangle $BCG=\frac{1}{2}\sqrt{(a^2+b^2)}(R-a-c)\sin\gamma-\frac{1}{2}ab$.

$\therefore$ Area$=\frac{1}{4}\pi(3R^2-2Rc+c^2)+\frac{1}{2}(R-a-c)^2(\pi-\gamma-\beta')+\frac{1}{2}(R-b+c)^2(\beta-\beta')+\frac{1}{2}\sqrt{(a^2+b^2)}(R-a-c)\sin\gamma-\frac{1}{2}ab$.

II. Solution by C. C. BEBOUT, Professor of Mathematics in Elgin High School, Elgin, Ill.; CHARLES C. CROSS, Libertytown, Md., and ELMER SCHUYLER, High Bridge, N. J.

As shown by the figure, the area grazed over is made up of one semi-circle, two quadrants, two sectors, and a triangle. We must sum the areas of these.

Area of semi-circle $HPK=\frac{1}{2}\pi R^2$ .....(1).

Area of quadrant $HAK=\frac{1}{4}\pi(R-c)^2$ .....(2).

Area of quadrant $KBI=\frac{1}{4}\pi(R-b+c)^2$ ..... (3).

By using formula $A=\sqrt{[s(s-a)(s-b)(s-c)]}$ we get area of triangle DLC $=\sqrt{[c(R-a)(R-a-b)(b-c)]}$ .....(4).

To find the areas of the sectors we must find the angles TDL and ICL. These angles are respectively the complements of angles LDC and DCL, which may be found as angles of the triangle LCD, whose sides are known. To find these angles use the trigonometric formula:

$$\tan\tfrac{1}{2}A=+\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Then $\tan\frac{1}{2}\angle LDC=\sqrt{\dfrac{c(R-a-b)}{(R-a)\,(b-c)}}$ and $\tan\frac{1}{2}\angle LCD=\sqrt{\dfrac{(R-a-b)(b-c)}{c(R-a)}}$

$$\therefore\ \angle LDT=(90^\circ-\angle LDC)=90^\circ-2\tan^{-1}\sqrt{\frac{c(R-a-b)}{(R-a)\,(b-c)}},$$

$$\text{and } \angle LCI=(90^\circ-\angle LCD)=90^\circ-2\tan^{-1}\sqrt{\frac{(R-a-b)(b-c)}{c(R-a)}}.$$

$$\therefore \text{ Area of sector } LDT=\frac{360^\circ}{90^\circ-2\tan^{-1}\sqrt{\frac{c(R-a-b)}{(R-a)(b-c)}}}[\pi(R-a-c)^2]\ldots\ldots(5),$$

$$\text{and area of sector } LCI=\frac{360^\circ}{90^\circ-2\tan^{-1}\sqrt{\frac{(R-a-b)(b-c)}{c(R-a)}}}[\pi(R-a-b+c)^2]\ldots(6).$$

Summing (1), (2), (3), (4), (5), and (6), we get the area grazed over,

$$\tfrac{1}{4}\pi(4R^2+b^2+2c^2-2bR-2bc)+\sqrt{}[c(R-a)(R-a-b)(b-c)]$$

$$+\frac{360^\circ}{90^\circ-\tan^{-1}\sqrt{\frac{c(R-a-b)}{(R-a)(b-c)}}}[\pi(R-a-c)^2]$$

$$+\frac{360^\circ}{90^\circ-2\tan^{-1}\sqrt{\frac{(R-a-b)(b-c)}{c(R-a)}}}[\pi(R-a-b+c)^2].$$

Solutions of problem 106 were received from P. S. Berg and Elmer Schuyler, and of problem 105 from Sylvester Robins. These solutions came too late for credit in last issue.

NOTE ON THE CALCULATION OF INTEREST AND DISCOUNT.

BY JOSEPH V. COLLINS, PH. D., PROFESSOR OF MATHEMATICS, STATE NORMAL SCHOOL, STEVENS POINT, WIS.

In the January number of the MONTHLY, Hon. J. H. Drummond, after giving the answer to the problem, 'What are the proceeds of a note discounted at a bank for 10 years at 10 per cent.,' as *nothing*, says, "The method of calculating discount used by banks was invented to evade the usury laws. I have thought that the court which first sustained the method could not have been well versed in mathematical principles."

A curious thing and commentary on the preceding is that all methods of calculating interest and discount are open to objection. Custom requires that interest be paid at the end of each year or specified fraction of a year. When it is not paid until the end of a period of two or more years, the lender is defrauded of the interest on his interest. If all interest payments are made promptly, it is equivalent to paying compound interest. The business world recognizes that compound interest is the only fair kind. Thus tables of bond values and the like are always made on a compound interest basis. But simple interest is

the only kind which can be collected unless annual or compound is called for in the contract. The courts have decided, probably because compound interest piles up so rapidly, that borrowers shall not be compelled to pay it unless they have agreed to. It is evident that if simple interest is unfair to a lender, so likewise is its analogue in discount, viz., true discount. In bank discount the bank not only collects a certain rate of interest on the loan, but also *on its own pay*, which is theoretically an absurd proceeding.

In partial payments we find the same difficulties presenting themselves. If payments are made within a year, according to the United States rule interest is collected and set at interest before it is due. The old so-called Connecticut rule tried to avoid this unfairness, but in so doing became too complicated for general use. Business men see that the mercantile rule is the only one which is fair to both parties when the whole transactions falls within a year. But the mercantile rule works injustice if the period covers more than a year, since by it the lender gets only simple interest, whereas the United States rule gives him a form of compound interest. The United States rule works injustice to the borrower whenever he pays less than the interest. Thus it is evident that the element of time and certain practical considerations have a great deal to do towards determining the appropriate method of counting interest.

Looked at from a practical standpoint it is easy to see why the banker collects bank instead of true discount. True discount requires a long division after a preliminary calculation. Tables could not be made for computing true discount which would be at all convenient to use. The great bulk of the loans made by banks are for less than 3 mouths. The difference between the bank and true discount of say $500 for 30 days is only a little over one cent. It would be worth more than 5 cents of the cashier's time to make the longer calculation. Of course the difference would not be so slight in every case, but it should be noted that one and sometimes more than one other person's time besides the cashier's is involved. Then liability to error is much greater in the longer calculation, and this is an important item. Hence it is plain that the method of discounting notes pursued by the banks was not adopted (as the writer once thought before he began making computations like the above) with the object of extorting more money from their customers, but for purely practical reasons.

ALGEBRA.

92. Proposed by W. F. BRADBURY, A. M., Head Master Latin School, Cambridge, Mass.

Find the sum to n terms of $1+3^3+5^3+\ldots$. [From Charles Smith's *Elementary Algebra*, page 403].

I. Solution by DR. E. D. ROE, Jr., Associate Professor of Mathematics in Oberlin College, P. O., Norwood, Mass.

We have $s_{3,n}=1^3+2^3+\ldots n^3=\left(\frac{n(n+1)}{2}\right)^2$. (Todhunter's *Algebra*, page 263).

Now $s_{3,2n}=2^3 s_{3,n}+[1^3+3^3+5^3+\ldots.(2n-1)^3]$.
Hence $1^3+3^3+5^3+\ldots.(2n-1)^3=s_{3,2n}-8s_{3,n}=n^2(2n^2-1)$.

II. Solution by J. OWEN MAHONEY, B. E., M. Sc., Master and Instructor in Mathematics and Science, Cooper Training School, Carthage, Tex.; COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; and J. SCHEFFER, A. M., Hagerstown, Md.

Let $1^3+3^3+5^3+\ldots.(2N-1)^3=A+BN+CN^2+DN^3+EN^4$.
Then $(2N+1)^3=B+(2N+1)C+D(3N^2+3N+1)+E(4N^3+6N^2+4N+1)$.
Equating coefficients, we find $E=2$, $D=0$, $C=-1$, $B=0$.
Therefore $S=A-N^2+2N^4$.
But when $N=1$, $A=0$. $\therefore S=N^2(2N^2-1)$.

III. Solution by T. W. PALMER, A. M., Professor of Mathematics, University of Alabama, University, Alabama; G. B. M. ZEER, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; CHARLES C. CROSS, Libertytown, Md.; J. SCHEFFER, A. M., Hagerstown, Md.; and ELMER SCHUYLER, A. M., High Bridge, N. J.

The general term of the series is $(2n-1)^3$.

$$\therefore \Sigma(2n-1)^3=8\Sigma n^3-12\Sigma n^2+6\Sigma n-\Sigma 1$$

$$=8\frac{n^2(n+1)^2}{4}-12\frac{n(n+1)(2n+1)}{6}+6\frac{n(n+1)}{2}-n=n^2(2n^2-1).$$

IV. Solution by J. SCHEFFER, A. M., Hagerstown, Md., and ELMER SCHUYLER, High Bridge, N. J.

From the well known formula $1^3+2^3+3^3+\ldots.n^3=\frac{n^2(n+1)^2}{4}$ we have

$$1^3+2^3+3^3+4^3+\ldots.(2n-2)^3+(2n-1)^3=(2n-1)^2n^2.$$

Denoting the required sum by S, we have,

$$S+2^3+4^3+6^3+\ldots.(2n-2)^3=(2n-1)^2n^2,$$

$$\text{or } S+2^3[1+2^3+3^3+\ldots.(n-1)^3]=(2n-1)^2n^2,$$

$$\text{or } S+2^3\frac{(n-1)^2n^2}{4}=(2n-1)^2n^2, \text{ whence } S=n^2(2n^2-1).$$

V. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.; O. S. WESTCOTT, A. M., North Division High School, Chicago, Ill.; ELMER SCHUYLER, High Bridge, N. J.; A. H. BELL, Hillsboro, Ill.; JOSIAH H. DRUMMOND, LL. D., Portland, Me.; P. S. BERG, Principal of Schools, Larimore, N. D.; W. L. HARVEY, Portland, Me.; CHARLES E. MEYERS, Canton, O.; ALOIS F. KOVARIK, Decorah Institute, Decorah, Ia.; HAROLD C. FISKE, Union College, Schenectady, N. Y.; and J. SCHEFFER, A. M., Hagerstown, Md.

Solving by the Differential Method, the general formula for the sum of the n terms of a series is

$$Sum=na+\frac{n(n-1)}{2}a_1+\frac{n(n-1)(n-2)}{2\times 3}a_2+\frac{n(n-1)(n-2)(n-3)}{2\times 3\times 4}a_3+\ldots\ldots$$

in which a=first term of series, and a_1, a_2, a_3, a_4, etc., are the respective first terms of the successive orders of differences.

Then 1 27 125 343 729....$(2n-1)^3$=given series.
26 98 218 386..... ....=first order of differences.
72 120 168 ....=second order of differences.
48 48 .....=third order of differences.
0.....=fourth order of differences.

Therefore $a=1$, $a_1=26$, $a_2=72$, $a_3=48$, $a_4=0$.

Substituting these values in the general formula, we obtain

$$Sum=n+13n(n-1)+12n(n-1)(n-2)+2n(n-1)(n-2)(n-3)$$
$$=2n^4-n^2=n^2(2n^2-1)$$

=the sum of the cubes of the first n odd numbers.

When $n=4$, $n^2(2n^2-1)=496$.

When $n=5$, $n^2(2n^2-1)=1225=\square$.

COROLLARIES BY M. A. GRUBER.

COROLLARY 1. In a similar manner we find $2^3+4^3+6^3....+(2n)^3=2n^2(n+1)^2$=*the sum of the cubes of the first n even numbers.*

COROLLARY 2. By a similar process we obtain $1+2^3+3^3+4^3+....+n^3=\left[\frac{n(n+1)}{2}\right]^2$; or *the sum of the cubes of the first n natural numbers is equal to the square of the sum of the numbers.* For $1+2+3+4+....+n=\frac{n(n+1)}{2}$.

COROLLARY 3. $1+3^3+5^3+... +(2n-1)^3=n^2(2n^2-1)=\square$, when $n=1$, 5, 29, 169, 985, 5741, etc., or when n=the integral hypothenuse of a right triangle whose legs are consecutive integers. [See AMERICAN MATHEMATICAL MONTHLY, Vol. IV., No. 1, pages 24-27].

COROLLARY 4. $2^3+4^3+6^3+....+(2n)^3$ can never be a square; for $2n^2(n+1)^2$ is *twice a square.*

GEOMETRY.

109. Proposed by CHARLES CARROLL CROSS, Libertytown, Md.

Two circles, radii in ratio 3:1, centers A and O_1 respectively, are drawn tangent externally to each other and internally to a given circle O, and on the same diameter; O_2 and O_2' are drawn tangent externally to O and internally to A and O_1; O_3 and O_3' are drawn tangent internally to O and externally to A and O_2; O_3 and O_3' are drawn tangent internally to O and externally to A and O_2, A and O_2', respectively, and so on. Prove O_4, O, O_4'; O_5, A, O_5'; O_9, A, O_3' and O_{10}, O, O_2' are collinear. [The letters apply to the centers of the circles].

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Construct the figure as indicated by the problem, and let B be the point of tangency of the circles A and O.

We will solve generally by taking $m:1$ as the ratio of the radii of circles A and O_1.

Put m=radius of circle A. Then 1=radius of circle O_1, $m+1$=radius of circle O, and $\frac{m(m+1)}{m(m+1)+(n-1)^2}$=radius of circle O_n, in which n=the number of the smaller circle, beginning with circle O_1.

(a). In $\triangle AO_nO$, taking AO as base,

$$AO_n=m+\frac{m(m+1)}{m(m+1)+(n-1)^2}=\frac{m[(m+1)^2+(n-1)^2]}{m(m+1)+(n-1)^2},$$

$$OO_n=m+1-\frac{m(m+1)}{m(m+1)+(n-1)^2}=\frac{(m+1)[m^2+(n-1)^2]}{m(m+1)+(n-1)^2},$$

$$AO=1,\ \text{area}=\frac{m(m+1)(n-1)}{m(m+1)+(n-1)^2},\ \text{and altitude}=\frac{2m(m+1)(n-1)}{m(m+1)+(n-1)^2}.$$

(b). We shall now find the value of n when centers O_n, O and O_r' are collinear.

Then $\angle O_r'OO_1=\angle O_nOB$; also OO_r' : sine $\angle O_r'OO_1=OO_n$: sine $\angle O_nOB$.

By substituting values as found in (a), and by using r and n in the proper places, we have

$$\frac{(m+1)[m^2+(r-1)^2]}{m(m+1)+(r-1)^2} : \frac{2m(m+1)(r-1)}{m(m+1)+(r-1)^2}$$

$$=\frac{(m+1)[m^2+(n-1)^2]}{m(m+1)+(n-1)^2} : \frac{2m(m+1)(n-1)}{m(m+1)+(n-1)^2}.$$

Whence $m^2+(r-1)^2 : r-1=m^2+(n-1)^2 : n-1$;

$$m^2+(r-1)^2 : (n-1)^2-(r-1)^2=r-1 : n-r;$$

$$m^2+(r-1)^2 : n+r-2=r-1 : 1.$$

From this we find $n=\frac{m^2}{r-1}+1$; also $r=\frac{m^2}{n-1}+1$.

When $r=n$, $(n-1)^2=m^2$; whence $n=m+1$.

$r=1$, $n=\infty$.

$r=2$, $n=m^2+1$.

$r=3$, $n=\frac{m^2}{2}+1$, m=*even* number.

$r=4$, $n=\frac{m^2}{3}+1$, m=multiple of 3.

$r=5$, $n=\frac{m^2}{4}+1$, m=*even* number.

etc., etc.

(*c*). We shall next find the value of n when centers O_n, A and O_r' are collinear.

Following a similar reasoning and substitution as in (*b*), we obtain

$$n=\frac{(m+1)^2}{r-1}+1 \text{ ; also } r=\frac{(m+1)^2}{n-1}+1.$$

When $r=n$, $(n-1)^2=(n+1)^2$; whence $n=m+2$.

$r=1$, $n=\infty$.

$r=2$, $n=(m+1)^2+1$.

$r=3$, $n=\frac{(m+1)^2}{2}+1$, m=*odd* number.

$r=4$, $n=\frac{(m+1)^2}{3}+1$, $m=3p-1$, as 2, 5, 8, 11, etc.

$r=5$, $n=\frac{(m+1)^2}{4}+1$, m=*odd* number.

etc., etc.

From (*b*) and (*c*) we obtain the following groups of collinear centers, when $m=3$: O_1, O, A, O_∞; O_2', O, O_{10}; O_2', A, O_{17}; O_3', A, O_9; O_4', O, O_4; and O_5', A, O_5.

It will be observed that where a center, in any of the above groups, has a *prime* mark, the mark may be transferred from the first to the last center of the group; as, for group O_2', O, O_{10}, we have O_2, O, O_{10}'. This is evident from an inspection of the figure.

COROLLARY 1. The figure presents *three varieties of triangles*, which, in general forms, may be represented as AO_nO, $AO_{n+1}O_n$, and $OO_{n+1}O_n$, whose *areas are rational*. Area of triangle AO_nO is shown in (*a*).

COROLLARY 2. The curve passing through the centers of the smaller circles, (*i. e.* through O_1, O_2, O_3.... $O_\infty[=B]$.... O_3', O_2', O_1), is an ellipse whose foci are A and O. For, $OO_1+AO_1=OO_2+AO_2=OO_n+AO_n=O_1B$.

The general values of the focal radii and ordinate of center O_r are shown in (*a*).

[NOTE. Mr. Charles C. Cross furnished me with the value $\frac{3}{13+n(n-2)}$ =radius of circle O_n, which I found correct according to his notation, and from which I deduced the general value $\frac{m(m+1)}{m(m+1)+(n-1)^2}$.]

Excellent solutions with diagrams were received from *G. B. M. ZERR*, *CHARLES C. CROSS*, and *J. SCHEFFER*.

DISCUSSION OF INVERSE FUNCTIONS.

BY COOPER D. SCHMITT, A. M., PROFESSOR OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENN.

1°. NOTATION. If $\sin x=a$, we know that x can be found, and hence in order to express this value of x explicitly we write it $x=\sin^{-1}a$. The first expression means, the sine of the arc, x, is a, and "sine" is the subject, that is, our attention is called to that as being equal to a. The second expression has the arc, x, for its subject, and the second member of the equation is algebraic short hand, telling what arc we mean.

We read it, "x equals the arc whose sine is a," or more briefly, "arc-sine a," *i. e.* x equals arc-sine a.

The expression is sometimes read "x equals the inverse sine of a," or "the anti-sine of a."

Similarly, $\tan 45°=1$, becomes $45°=\frac{1}{4}\pi=\tan^{-1}1$,

$\cos 60°=\frac{1}{2}$, becomes $60°=\frac{1}{3}\pi=\cos^{-1}\frac{1}{2}$.

Also, $y=\tan^{-1}x$ can be written $\tan y=x$, and so on. The convention is, that what is inverse on one side is written direct on the other; thus $\sec^{-1}a=x$ becomes $\sec x=a$.

2°. Since $\tan x=\tan x$ we can say $\tan^{-1}(\tan x)=x$ or $\tan(\tan^{-1})x=x$. This becomes self-evident when expressed at length. Thus, the tangent of the arc whose tangent is x is x.

Similarly, $\cos^{-1}(\cos a)=a$, $\sin\sin^{-1}b=b$. The two symbols are not said to cancel each other but to annul or neutralize each other.

3°. There are always two angles less than 360° whose functions equal a certain quantity, but for simplicity, we will always mean the smaller of the two. Thus $\tan^{-1}1=45°$ or 225°, since the tangent of either of these angles is 1. But 45° is taken unless for some special reason the larger angle is desired.

4°. Any inverse function can be converted into all the other inverse functions.

Thus $\sin^{-1}\frac{3}{5}$ becomes $\tan^{-1}\frac{3}{4}$ by constructing a triangle with hypotenuse 5 and perpendicular 3. From this we see that $\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}=\cos^{-1}\frac{4}{5}=\cot^{-1}\frac{4}{3}$, etc., etc.

Similarly $\cos^{-1}\frac{2}{3}=\sin^{-1}\frac{\sqrt{5}}{3}=\tan^{-1}\frac{\sqrt{5}}{2}$, etc., etc.

Similarly $\tan^{-1}1\frac{5}{12}=\cot^{-1}1\frac{2}{5}=\sin^{-1}1\frac{5}{13}=\cos^{-1}1\frac{12}{13}$.

5°. Any formula of trigonometry expressed in direct notation can be converted into a corresponding formula in the inverse notation.

Thus to convert $\sin 2x=2\sin x\cos x$ ...(1).

Let $\sin x=a$. Then $\cos x=\sqrt{1-a^2}$ and $x=\sin^{-1}a$.

(1) can be written according to the convention agreed upon in 1°.

$2x=\sin^{-1}(2\sin x\cos x)$. Whence $2\sin^{-1}a=\sin^{-1}(2a\sqrt{1-a^2})$.

Similarly $\cos(x+y)=\cos x\cos y-\sin x\sin y$....(2).

Let $\cos x=a$, then $x=\cos^{-1}a$. Let $\cos y=b$, then $y=\cos^{-1}b$.

And (2) can be written $x+y=\cos^{-1}(\cos x\cos y-\sin x\sin y)$.

Whence $\cos^{-1}a+\cos^{-1}b=\cos^{-1}(ab+\sqrt{1-a^2}\sqrt{1-b^2})$....(3), which is a formula for adding two inverse cosines and getting the result as a single inverse cosine.

Again $\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$ or $x+y=\tan^{-1}\left(\dfrac{\tan x+\tan y}{1-\tan x\tan y}\right)$ ...(4).

Let $\tan x=a$, $\tan y=\tan b$, and (4) becomes

$$\tan^{-1}a+\tan^{-1}b=\tan^{-1}\frac{a+b}{1-ab}\ldots.(5).$$

Similarly, $\tan^{-1}a-\tan^{-1}b=\tan^{-1}\dfrac{a-b}{1+ab}$....(6).

6°. From 4° we see that any inverse function can be expressed as an inverse tangent, and by these last two formulae, (5) and (6), we can add or subtract any two inverse tangents; hence these two formulae are all that are necessary for combining any number of inverse functions.

Thus, $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\dfrac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{91}}=\tan^{-1}\frac{2}{9}$;

$$\tan^{-1}\tfrac{3}{4}+\tan^{-1}\tfrac{4}{3}=\tan^{-1}\frac{\frac{3}{4}+\frac{4}{3}}{1-1}=\tan^{-1}\infty=90°=\tfrac{1}{2}\pi\ ;$$

$$2\tan^{-1}\tfrac{3}{4}=\tan^{-1}\tfrac{3}{4}+\tan^{-1}\tfrac{3}{4}=\tan^{-1}\frac{\frac{3}{4}+\frac{3}{4}}{1-\frac{9}{16}}=\tan^{-1}\tfrac{24}{7}.$$

To add $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}$ we might use (3), but it is better to convert the $\cos^{-1}$ into $\tan^{-1}$ as in 4°.

Thus $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{5}{12}=\tan^{-1}\dfrac{\frac{3}{4}+\frac{5}{12}}{1-\frac{5}{16}}=\tan^{-1}\frac{56}{33}$.

7°. If fractional coefficients occur they must be gotten rid of.

Thus $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}-\frac{1}{2}\cos^{-1}\frac{3}{5}$ becomes

$$\tfrac{1}{2}(2\tan^{-1}\tfrac{1}{4}+2\tan^{-1}\tfrac{2}{9}-\cos^{-1}\tfrac{3}{5})$$

$$=\tfrac{1}{2}(\tan^{-1}\tfrac{1}{4}+\tan^{-1}\tfrac{1}{4}+\tan^{-1}\tfrac{2}{9}+\tan^{-1}\tfrac{2}{9}-\tan^{-1}\tfrac{4}{3})$$

$$=\tfrac{1}{2}(\tan^{-1}\frac{\frac{1}{4}+\frac{1}{4}}{1-\frac{1}{16}}+\tan^{-1}\frac{\frac{2}{9}+\frac{2}{9}}{1-\frac{4}{81}}-\tan^{-1}\tfrac{4}{3})$$

$$=\tfrac{1}{2}(\tan^{-1}\tfrac{8}{15}+\tan^{-1}\tfrac{36}{77}-\tan^{-1}\tfrac{4}{3})$$

$$=\tfrac{1}{2}(\tan^{-1}\frac{\frac{8}{15}+\frac{36}{77}}{1-\frac{8}{15}\cdot\frac{36}{77}}-\tan^{-1}\tfrac{4}{3})=\tfrac{1}{2}(\tan^{-1}\tfrac{1456}{1155}-\tan^{-1}\tfrac{4}{3})$$

$$=\tfrac{1}{2}(\tan^{-1}\tfrac{4}{3}-\tan^{-1}\tfrac{4}{3})=0.$$

$4\tan^{-1}\frac{1}{5}$ can be written $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{5}$ and now the addition formulae can be used at once.

Similarly, for any integral coefficients.

Suppose we are to show that $\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\sin^{-1}\frac{3}{5}$.

We proceed as follows: $\cos^{-1}\frac{63}{65}=\tan^{-1}\frac{16}{63}$, $\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$, and now we have to show that $\tan^{-1}\frac{16}{63}+2\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{3}{4}$, which is done at once as in previous example.

It is thus seen that the formulae $\tan^{-1}a\pm\tan^{-1}b=\tan^{-1}\frac{a\pm b}{1\mp ab}$ will apply to most if not all of the examples as given in our text-books and that the work is done entirely in the inverse notation.

CALCULUS.

84. Proposed by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Find the equation to the curve upon which a given ellipse must roll in order that one of its foci may describe a straight line.

No solution of this problem has yet been received.

It can be easily shown that $ds=(1+\frac{\rho}{\rho'})ds'$ where ρ and ρ' are the radii of curvature of the rolling and fixed curves, and ds and ds' arcs of the roulette (the curve generated by the focus of the ellipse), and the pedal curve, the origin of which is the generating point. Finding expressions for ds and ds' and substituting in the above equation and solving for ρ, we have an expression for the curvature of the fixed curve. By substituting for ρ, the general value for the radius of curvature, we derive a very complicated differential equation of the curve. I am not aware that the problem has ever been solved. If any of our contributors will send us a complete solution we shall be pleased to publish it. ED. F.

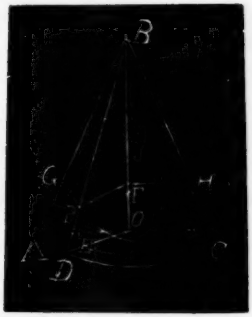
85. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A line of double curvature, beginning at some point in the circumference of the base circle of a right cone, winds itself under the constant inclination β to the base circle around the curved surface of the cone. Find its length and that of its projection upon the base circle.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let ABC be the cone, CPG the line of curvature, σ its length, CE its projection on base, s the length of this projection, BO the z-axis. Also let $DO=CO=a$, $PF=r$, $\angle DPE$=semi-vertical angle of cone$=\gamma$, $\angle PCD=\beta$. Let P, C be

two consecutive points on the curve. Draw PE perpendicular to DO. Then $PC=d\sigma$, $CE=ds$, $PE=dz$, $DE=dr$, $\angle COE=d\theta$, $\angle DPE=\angle DBO=\gamma$.



Then $DE=PD\sin\gamma=PC\sin\gamma\sin\beta$.

$\therefore\ dr=\sin\gamma\sin\beta d\sigma$.

$\therefore\ \sigma=\dfrac{r}{\sin\gamma\sin\beta}=\dfrac{a}{\sin\gamma\sin\beta}$, where $a=r$.

$PE=PD\cos\gamma=PC\cos\gamma\sin\beta$.

$\therefore\ dz=\cos\gamma\sin\beta d\sigma$. $\therefore\ z=\sigma\cos\gamma\sin\beta$.

$CE=\sqrt{(CP^2-PE^2)}$.

$\therefore\ ds=\sqrt{(d\sigma^2-dz^2)}=d\sigma\sqrt{(1-\cos^2\gamma\sin^2\beta)}$.

$\therefore\ s=\sigma\sqrt{(1-\cos^2\gamma\sin^2\beta)}=\dfrac{a\sqrt{(1-\cos^2\gamma\sin^2\beta)}}{\sin\gamma\sin\beta}$

$r.PFH=PH=PC\cos\beta$.

$\therefore\ rd\theta=\cos\beta d\sigma$.

$\therefore\ d\theta=\dfrac{\cos\beta d\sigma}{r}=\dfrac{\cot\beta}{\sin\gamma}.\dfrac{d\sigma}{\sigma}$. $\therefore\ \theta=\dfrac{\cot\beta}{\sin\gamma}\log\sigma$.

$x=r\cos\theta$, $y=r\sin\theta$, $z=\sigma\cos\gamma\sin\beta=r\cot\gamma$ are the equations to the curve.
σ and s as given above are the values asked for in the problem.

MECHANICS.

77. Proposed by ELMER SCHUYLER, High Bridge, N. J.

At what elevation must a shell be projected with a velocity of 400 feet that it may range 7500 feet on a plane which descends at an angle of 30°?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; J. SCHEFFER, A. M., Hagerstown, Md., and the PROPOSER.

Let $OR=7500$ feet$=R$, $V=400$ feet, $\angle POS=e=$angle of elevation, and $\angle ROS=i=30°$.

$\therefore\ PR : OP=\sin(e+i) : \cos i$.

But $OP=vt$, and $PR=\frac{1}{2}gt^2$.

$\therefore\ t=2v\sin(e+i)/g\cos i$. $OR : PR=\cos e : \sin(e+i)$.

$\therefore\ R=gt^2\cos e/2\sin(e+i)=2v^2\sin(e+i)\cos e/g\cos^2 i$.

$$\therefore\ \tan e=\frac{v^2\pm\sqrt{(2v^2gR\sin i-g^2R^2\cos^2 i+v^4)}}{gR\cos i}=\frac{160000\pm 143348.62575}{208928.62775}.$$

$\therefore\ \tan e=1.451925$ or $.079699$.

$\therefore\ e=55°\ 26'\ 36''$ or $4°\ 33'\ 24''$.

II. Solution by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering and Physics, Agricultural and Mechanical College, College Station, Texas.

If the point of projection be origin and the path be regarded as a parabola its equation will be

$$y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha} \ldots (1),$$

where α=angle of elevation.

At the point where the shell is to strike the plane, $y=-3750$ feet and $x=6495$ feet. Inserting these values in equation (1) there results

$$-3750\times 400\cos^2\alpha = 6495\times 200\times \sin\alpha\cos\alpha - (1299)^2.$$

For $\cos^2\alpha$ write $1+\cos 2\alpha$, and for $2\sin\alpha\cos\alpha$ write $\sin 2\alpha$, and as $\sin 2\alpha = \sqrt{(1-\cos^2 2\alpha)}$, we get

$$\sqrt{(1-\cos^2 2\alpha)} = .722 - .577\cos 2\alpha \ldots . (2).$$

Solving (2) we get $\cos 2\alpha = .314 \pm .676$.

$\therefore 2\alpha = 8^\circ\ 48'$ or $111^\circ\ 12'$ and $\alpha = 4^\circ\ 24'$ or $55^\circ\ 36'$, which values satisfy the condition

$$\alpha'' - \tfrac{1}{2}(\tfrac{1}{2}\pi - 30) = \tfrac{1}{2}(\tfrac{1}{2}\pi - 30) - \alpha'.$$

See Tait and Steele's *Dynamics of a Particle*, page 90.

Also solved by *P. H. PHILBRICK*.

78. Proposed by ALOIS F. KOVARIK, Professor of Mathematics, Decorah Institute, Decorah, Iowa.

A cone and a cylinder having equal heights and equal circular bases are filled with water; if they have equal holes in the bases, respectively, how many times as long will it take the cylinder to empty as the cone?

I. Solution by P. H. PHILBRICK, C. E., Lake Charles, La.

Let r=radius of base, h=altitude, k=area of orifice, and x=height of the water at the time t.

I. For the cylinder, the discharge in time dt is, $k\sqrt{(2gx)}dt$; and since in the same time the surface of the water descends a distance dx, the quantity in the vessel is *lessened* $\pi r^2 dx$.

$\therefore k\sqrt{(2gx)}dt = \pi r^2 dx$, and

$$t = -\frac{\pi r^2}{k\sqrt{(2g)}}\int \frac{dx}{\sqrt{x}} = -\frac{2\pi r^2}{k\sqrt{(2g)}}x^{\frac{1}{2}} + c = \frac{2\pi r^2}{k\sqrt{(2g)}}(h^{\frac{1}{2}} - x^{\frac{1}{2}}).$$

II. For the cone, we have $y=(r/h)(h-x)$, and the area of section

$$=\pi y^2 = \frac{\pi r^2}{h^2}(h-x)^2.$$

$$\therefore k\sqrt{(2gx)}dt=\frac{\pi r^2}{h^2}(h-x)^2dx, \text{ and } t=-\frac{\pi r^2}{h^2 k\sqrt{(2g)}}\int\frac{(h-x)^2}{x^{\frac{1}{2}}}dx$$

$$=-\frac{\pi r^2}{h^2 k\sqrt{(2g)}}\int\frac{(h^2-2hx+x^2)dx}{x^{\frac{1}{2}}}$$

$$=-\frac{\pi r^2}{h^2 k\sqrt{(2g)}}\cdot(2h^2x^{\frac{1}{2}}-\frac{4h}{3}x^{\frac{3}{2}}+\tfrac{2}{5}x^{\frac{5}{2}})+c.$$

But $t=0$ for $x=h$. $\therefore c=\frac{\pi r^2}{h^2 k\sqrt{(2g)}}\,\frac{16}{15}h^{\frac{1}{2}}$.

Hence $t=-\frac{\pi r^2}{h^2 k\sqrt{(2g)}}(2h^2x^{\frac{1}{2}}-\frac{4h}{3}x^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}})+\frac{\pi r^2}{k\sqrt{(2g)}}\,\frac{16}{15}h^{\frac{1}{2}}$.

For $x=0$, the time of emptying vessel$=t_2=\frac{\pi r^2}{k\sqrt{(2g)}}\cdot\frac{16}{15}h^{\frac{1}{2}}$.

This is $\frac{8}{15}$ of the time of emptying the cylinder.

II. Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Let A=the area of the descending surface, O=the area of the orifice, and x=the depth of the water at the end of any time t.

The quantity of water discharged through the orifice in the infinitely small time dt is $O.dt(2gx)^{\frac{1}{2}}$, the velocity of discharge being $(2gx)^{\frac{1}{2}}$; but in the same time the surface has descended through the distance dx and the quantity discharged is Adx.

$\therefore O.dt(2gx)^{\frac{1}{2}}=Adx$, or $dt=\frac{Adx}{O.(2gx)^{\frac{1}{2}}}$, as general formula for any shape of vessel.

Now for the cylinder $A=\pi r^2$ and therefore $dt=-\frac{\pi r^2 dx}{O(2gx)^{\frac{1}{2}}}$ and

$$t=-\frac{\pi r^2}{O(2g)^{\frac{1}{2}}}\int_h^0\frac{dx}{x^{\frac{1}{2}}}=\frac{2\pi r^2 h^{\frac{1}{2}}}{O(2g)^{\frac{1}{2}}}=\frac{\pi r^2(2h)^{\frac{1}{2}}}{O.g^{\frac{1}{2}}};$$

and for the cone $A=\frac{\pi r^2(h-x)^2}{h^2}$ and

$$\therefore dt=-\frac{\pi r^2(h-x)^2dx}{Oh^2(2g)^{\frac{1}{2}}} \text{ and } t=-\frac{\pi r^2}{Oh^2(2g)^{\frac{1}{2}}}\int_h^0\frac{(h-x)^2dx}{x^{\frac{1}{2}}}$$

$$\frac{\pi r^2}{Oh^2(2g)^{\frac{1}{2}}}\cdot\frac{16h^{\frac{5}{2}}}{15}=\frac{8}{15}\cdot\frac{\pi r^2(2h)^{\frac{1}{2}}}{O.g^{\frac{1}{2}}}.$$

Comparing the two results we see that the cone empties in $\frac{8}{15}$ of the time it takes the cylinder, or the cylinder takes $1\frac{7}{8}$ as long as the cone to empty. The minus sign is prefixed because x decreases as t increases.

Also solved by *G. B. M. ZERR, ELMER SCHUYLER, J. SCHEFFER,* and *J. C. NAGLE.*

NOTE. In reference to problem 63, Dr. Arnold Emch says: "I had the problem solved by my class in graphic statics by a purely graphical method and the following values (approximations) were obtained: $\angle ABE=46°$, $\angle BAD=56°$, tension in $BE=56.8$, tension in $AD=71$. This shows that the solution in the MONTHLY is correct."

DIOPHANTINE ANALYSIS.

71. Proposed by A. H. BELL, Hillsboro, Ill.

Find five numbers such that the product of any two plus 1 will equal a square.

III. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

By using $(s-1)^2$, the denominator of Euler's fifth number, where $s=4n(n-1)(n+1)[4n(2n-1)(2n+1)]$, I have found five numbers in terms of n: $x=n-1$, $y=n+1$, $z=4n$, $w=4n(2n-1)(2n+1)$, and

$$v=\frac{4n(2n-1)(2n+1)[2n(2n-1)-1][2n(2n+1)-1](8n^2-1)}{\{4n(n-1)(n+1)[4n(2n-1)(2n+1)]-1\}^2}.$$

The numerator of v is four times the product of the roots of the six squares $xy+1$, $xz+1$, $yz+1$, $xw+1$, $yw+1$, and $zw+1$.

The denominator of v is the square of $(xyzw-1)$.

Take $n=1, 2, 3, 4, 5, 6$, etc. We then obtain the following sets of five numbers:

$0, 2, 4, 12, 420$;

$1, 3, 8, 120, \frac{777480}{(2879)^2}$;

$2, 4, 12, 420, \frac{35455980}{(40319)^2}$;

$3, 5, 16, 1008, \frac{499902480}{(241919)^2}$;

$4, 6, 20, 1980, \frac{3822388020}{(950399)^2}$;

$5, 7, 24, 3432, \frac{20000100120}{(2882879)^2}$; etc., etc., etc.

We also find

$$xv+1=\frac{\{(2n-1)(2n+1)[2n(2n-1)-1][2n(2n+1)-1]-2n(8n^2-1)\}^2}{(s-1)^2};$$

$$yv+1=\frac{\{(2n-1)(2n+1)[2n(2n-1)-1][2n(2n+1)-1]+2n(8n^2-1)\}^2}{(s-1)^2};$$

$$zv+1=\frac{\{16n^4(2n-1)(2n+1)-(8n^2-1)\}^2}{(s-1)^2}; \text{ and}$$

$$wv+1=\frac{\{24n^2(2n^2-1)(2n-1)(2n+1)-1\}^2}{(s-1)^2}.$$

74. Proposed by O. W. ANTHONY, M. Sc., Instructor in Mathematics, Boys' High School, New York City.

Solve $x^2+y^2=\square$, $z^2+w^2=\square$, $y^2+w^2=\square$.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Take any two integral equations in which the sum of two squares equals a square, as

$$a^2+b^2=c^2, \text{ and } a_1^2+b_1^2=c_1^2.$$

Multiply the terms of the first equation by the first term and the second term, respectively, of the second equation. Also multiply the terms of the second equation by the first term and the second term, respectively, of the first equation. We then have

$$(aa_1)^2+(a_1b)^2=(a_1c)^2 \ldots . (1),$$
$$(ab_1)^2+(bb_1)^2=(b_1c)^2 \ldots . (2),$$
$$(aa_1)^2+(ab_1)^2=(ac_1)^2 \ldots . (3),$$
$$(a_1b)^2+(bb_1)^2=(bc_1)^2 \ldots \ (4).$$

Now put $x=aa_1$, $y=a_1b$, $z=ab_1$, and $w=bb_1$; then equations (1), (2), and (4) are the three required by the problem, there being added, in the solution, $x^2+z^2=\square \ldots . (3)$.

By means of the formula $(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2$, find a few integral numerical equations.

Take $m=2$, $n=1$; then $4^2+3^2=5^2 \ldots . (1)$.

Take $m=3$, $n=2$; then $12^2+5^2=13^2 \ldots . (2)$.

Take $m=4$, $n=1$; then $8^2+15^2=17^2 \ldots . (3)$.

Take $m=5$, $n=2$; then $20^2+21^2=29^2 \cdots . (4)$, etc.

From (1) and (2), $x=48$, $y=36$, $z=20$, $w=15$.

From (1) and (3), $x=32$, $y=24$, $z=60$, $w=45$.

From (1) and (4), $x=80$, $y=60$, $z=84$, $w=63$.

From (2) and (3), $x=96$, $y=40$, $z=180$, $w=75$, etc.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Take $(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2$,

and $(2pq)^2+(p^2-q^2)^2=(p^2+q^2)^2$,

two general equations for finding the sum of two squares equal to a square.

Multiply the terms of the first equation by the first term and second term, respectively, of the second equation. Also multiply the terms of the second equation by the first term and the second term, respectively, of the first equation. Then, $[4mnpq]^2+[2pq(m^2-n^2)]^2=[2pq(m^2+n^2)]^2$....(1).

$[2mn(p^2-q^2)]^2+[(m^2-n^2)(p^2-q^2)]^2=[(m^2+n^2)(p^2-q^2)]^2$....(2),

$[4mnpq]^2+[2mn(p^2-q^2)]^2=[2mn(p^2+q^2)]^2$....(3),

$[2pq(m^2-n^2)]^2+[(m^2-n^2)(p^2-q^2)]^2=[(m^2-n^2)(p^2+q^2)]^2$....(4).

Now put $x=4mnpq$, $y=2pq(m^2-n^2)$, $z=2mn(p^2-q^2)$, and $w=(m^2-n^2)(p^2-q^2)$, and the general equations (1), (2), and (4) are the three required by the problem; $x^2+z^2=\square$, as equation (3), being added in the solution.

m, n, p, and q are any integers, $m>n$, and $p>q$, and to obtain different values for x, y, z, and w, not more than two values, assigned to m, n, p, and q may be alike.

Take $m=2$, $n=1$, $p=3$, $q=2$; then $x=48$, $y=36$, $z=20$, $w=15$.

Take $m=2$, $n=1$, $p=3$, $q=1$; then $x=24$, $y=18$, $z=32$, $w=24$.

Take $m=4$, $n=3$, $p=3$, $q=2$; then $x=288$, $y=84$, $z=120$, $w=35$, etc.

III. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

y and w only, of the four quantities, are used twice. Assign to y a numerical value such that it is found at least in *two sets* of the sum of two squares equal to a square.

(A).....Put $y=8$. We find $8^2+6^2=10^2$, and $8^2+15^2=17^2$.

(1). Take $x=6$ and $w=15$.

We find $15^2+8^2=17^2$, $15^2+20^2=25^2$, $15^2+36^2=39^2$, and $15^2+112^2=113^2$. Then $z=20$, 36, or 112.

(B).....Put $y=12$. We find $12^2+5^2=13^2$, $12^2+9^2=15^2$, $12^2+16^2=20^2$, and $12^2+35^2=37^2$.

(1). Take $x=5$, and $w=9$.

We find $9^2+12^2=15^2$, and $9^2+40^2=41^2$. Then $z=40$.

(2). Take $x=5$, and $w=16$.

We find $16^2+12^2=20^2$, $16^2+30^2=34^2$, and $16^2+63^2=65^2$.

Then $z=30$ or 63.

(3). Take $x=5$, and $w=35$.

We find $35^2+12^2=37^2$, $35^2+84^2=91^2$, $35^2+120^2=125^2$, and $35^2+612^2=613^2$. Then $z=84$, 120, or 612.

(4). Take $x=9$, and $w=16$.

Then from (2), $z=30$, or 63. And so on.

IV. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

In problems where $x^2+y^2=\square$, $z^2+w^2=\square$, $x^2+z^2=\square$, and $y^2+w^2=\square$, we have the proportion $x : y=z : w$.

Now take two integers the sum of whose squares equals a square, and arrange them in an identical proportion.

Then take two integers of the same kind and arrange them, underneath the first proportion, in an identical proportion of alternation as compared with the first proportion.

Then find the products, term by term, of these two proportions ; and the four products will be the required numbers.

Take $3^2+4^2=5^2$, and $5^2+12^2=12^2$.

$$\begin{array}{c} x : y=z : w \\ \hline 3 : 4=3 : 4 \\ 3 : 5=12 : 12 \\ \hline 15 : 20=36 : 48 \end{array}$$

$15^2+20^2=25^2$, $36^2+48^2=60^2$, $15^2+36^2=39^2$, $20^2+48^2=52^2$.

V. Solution by J. H. DRUMMOND, LL. D., Portland, Me.

Manifestly x and y, and z and w, are the bases and perpendiculars of two different right-angled triangles. Hence $x=m^2-n^2$, and $y=2mn$; and $z=p(m^2-n^2)$, and $w=2pmn$. But $y^2+w^2=\square$. Or $4p^2m^2n^2+4m^2n^2=\square$, or $p^2+1=\square$=(say) $(pq-1)^2$. From which $p=\frac{2q}{q^2-1}$ Then $z=\frac{2q(m^2-n^2)}{q^2-1}$, and $w=\frac{2qmn}{q^2-1}$, in which m, n, and q may be any numbers, $q>1$, and $m>n$.

Also solved by *A. H. BELL, CHARLES C. CROSS, ELMER SCHUYLER*, and *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

112. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The cost of an article is $\$4.\frac{\dot{2}9\dot{7}}{1.003}$. The selling price is $\$6.\frac{1000}{333333}$. What is the gain % ?

113. Proposed by B. F. SINE, Principal of Normal School, Capon Bridge, W. Va.

In what time can a note of \$5280, bearing 6% interest, be paid by paying \$600 a year ? [Solve by arithmetic].

Solutions of these problems should be sent to B. F. Finkel not later than June 10.

ALGEBRA.

100. Proposed by W. H. CARTER, Vice President and Professor of Mathematics, Centenary College, Jackson, La.

Solve, $x^x+y=y^{4a}$, $y^x+y=x^a$.

101. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Prove that $(1+2+3+\ldots.+n)+\frac{n}{2!}(1^2+2^2+3^2+\ldots.+n^2)+\frac{n(n-1)}{3!}$

$(1^3+2^3+3^3+\ldots.+n^3)+\frac{n(n-1)(n-2)}{4!}(1^4+2^4+3^4+\ldots.+n^4)+\ldots\ldots\ldots$

$+\frac{n(n-1)(n-2)}{4!}(1^{n-3}+2^{n-3}+3^{n-3}+\ldots.+n^{n-3})+\frac{n(n-1)}{3!}(1^{n-2}+2^{n-2}+3^{n-2}$

$+\ldots.+n^{n-2})+\frac{n}{2!}(1^{n-1}\,2^{n-1}\,3^{n-1}+\ldots.+n^{n-1})+(1^n+2^n+3^n+\ldots.+n^n)$

$=(n+1)^n-1$, and substitute for $n=2$, 3, 4, 5, and 6.

*** Solutions of these problems should be sent to J. M. Colaw not later than June 10.

GEOMETRY.

120. Proposed by P. C. CULLEN, Principal Public Schools, Indianola, Neb.

Draw a circle tangent to a given circle and tangent to a given chord at a given point.

121. Proposed by AUGUSTUS J. REEF, Carbondale, Ind.

Construct a triangle having given its three medians. [From Wentworth's *Plane and Solid Geometry*].

122. Proposed by G. I. HOPKINS, A. M., Professor of Mathematics and Physics, Manchester High School, Manchester, N. H.

If perpendiculars are dropped from the vertices of a regular polygon upon any diameter of the circumscribed circle, the sum of the perpendiculars which fall on one side of this diameter is equal to the sum of those which fall on the opposite side. [From Chauvenet's *Treatise on Elementary Geometry*.

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CALCULUS.

90. Proposed by ELMER SCHUYLER, High Bridge, N. J.

Prove that the evolute of the logarithmic spiral is an equal logarithmic spiral. [From Byerly's *Integral Calculus*].

91. Proposed by GUY B. COLLIER, Schenectady, N. Y.

Find the area of a loop of the curve $r^2\cos\theta=a^2\sin 3\theta$. [From Hall's *Differential and Integral Calculus*].

*** Solutions of these problems should be sent to J. M. Colaw not later than June 10.

MECHANICS.

89. Proposed by GUY B. COLLIER, Schenectady, N. Y.

Assuming that the Northern Pacific R. R. tracks between Fargo and Bismark (North Dakota) to lie on the 47th parallel of latitude; also that the Limited Express weighs 300 tons, and that a speed of 60 miles per hour is maintained between the two places; find the difference between the vertical pressures on the rails of the Express east and the Express west.

90. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Adopting the hypothesis that the planets were originally all one mass revolving about a fixed center and were formed by an explosion of this mass at some point in its path; prove that, if the law of nature were that force varies directly as the distance, the planets would all have collided again simultaneously, and find an expression for the time between the explosion and collision.

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

DIOPHANTINE ANALYSIS.

80. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find three square numbers whose reciprocals form an arithmetical progression.

*** Solutions of this problem should be sent to J. M. Colaw not later than June 10.

AVERAGE AND PROBABILITY.

73. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

On an average 1 vessel out of every n is wrecked. Find the chance that out of m vessels expected p at least will arrive safely.

74. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

From a point in the circumference of a circular field a projectile is thrown at random with a given velocity which is such that the diameter of the field is equal to the greatest range of the projectile. Find the chance of its falling into the field. [From Byerly's *Integral Calculus*, page 209].

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

MISCELLANEOUS.

77. Proposed by T. E. COLE, Columbus, Ohio.

It is said that a base-ball pitcher throws curves. Give a scientific explanation of how it is done.

78. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

The center of a regular polygon of n sides moves along a diameter of a given circle, the plane of the polygon being perpendicular to the diameter, and its magnitude varying

in such a manner that one of its diagonals always coincides with a chord of the circle; find the surface and the volume generated, and thence deduce the formulae for the surface and the volume of a sphere.

*** Solutions of these problems should be sent to J. M. Colaw not later than June 10.

EDITORIALS.

Editor Finkel was elected a member of the London Mathematical Society on April 13th.

THE UNIVERSITY OF CHICAGO, SUMMER, 1899. The following Mathematical Courses will be offered: By Professor *Maschke*, Theory of Functions of a Complex Variable, Abstract Groups; Professor *Hathaway*, Quaternions, Plane Analytics; Assistant Professor *Young*, Conferences on the Pedagogy of Mathematics, Determinants; Assistant Professor *Skinner*, College Algebra; Dr. *Slaught*, Differential Equations, Differential Calculus; Dr. *Boyd*, Twisted Curves, Solid Geometry.

BOOKS.

An Introduction to the Differential and Integral Calculus and Differential Equations. By F. Granville Taylor, M. A., B. Sc., Mathematical Lecturer at University College, Nottingham. 8vo. Cloth, 592 pages. Price, 8s. London and New York: Longmans, Green & Co.

This work comprises three sections, in the first of which is given a very thorough treatment of the Differential Calculus and its applications; the second is devoted to treatment of the Integral Calculus; and the third deals with the elementary methods of solving Ordinary Differential Equations of the first and second orders. In the Differential and Integral Calculus, the author has given a few practical applications as early as possible, in order that the beginner may have some notion of the uses to which the Calculus may be put. Curve Tracing receives a good deal of attention; Hyperbolic Functions and their differentiation has received due consideration. Throughout the work, numerous examples are given, these being well selected and graded in a way to stimulate and inspire the student. The subject of the Calculus as presented in this work is clear and simple, and is a worthy rival of the many valuable works on this subject. B. F. F.

A Text-book of General Physics for the use of Colleges and Scientific Schools. By Charles S. Hastings, Ph. D., and Frederic E. Beach, Ph. D., of Yale University. 8vo. Half Leather Back, v+768 pages. Price, $2.95. Boston: Ginn & Co.

In this book, the fact is emphasized that a knowledge of Elementary Mechanics is the logical basis of the whole science of Physics. With this in view, we find here a more complete treatment of Mechanics than is ordinarily the case, especially in the physical notions which attach to the simplest cases of the action of force. Numerous problems are appended to the various chapters, the solutions of which will go far towards impressing the principles upon the mind of the student. The method of presenting the subject of Physics as here given is very good. B. F. F.

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ON SYMMETRIC FUNCTIONS.

By E. D. ROE, Jr., Associate Professor of Mathematics in Oberlin College.

[Continued from April Number.]

5. FORMULAS FOR THE FOUR KINDS OF REDUCTION.

We will place here together the four formulas of reduction. They are:

$$\begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} \dots\dots n^{\lambda_n} \\ 0^m \; \varkappa_1 \; \varkappa_2 \dots\dots \varkappa_n \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0 - \lambda} \; 1^{\lambda_1} \dots\dots \; n^{\lambda_n} \\ 0^{\kappa} \; \varkappa_1 \; \varkappa_2 \dots\dots \; \varkappa_n \end{pmatrix}, \quad m - \lambda = \kappa_1,$$

$$\begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} \dots\dots n^{\lambda_n} \\ 0^m \; \varkappa_1 \; \varkappa_2 \dots\dots \varkappa_n \end{pmatrix} = (-1)^{n\kappa_n} \begin{pmatrix} 0^{\lambda_0} \; 1^{\lambda_1} \dots\dots\dots\

1. The First Formula.

The first formula of derivation is

$$\begin{pmatrix} 0^{\lambda_0} & 1^{\lambda_1} & \dots\dots & n^{\lambda_n} \\ 0^m & \varkappa_1 & \varkappa_2 \dots\dots & \varkappa_n \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0+\lambda} & 1^{\lambda_1} & \dots\dots & n^{\lambda_n} \\ 0^{m+\lambda} & \varkappa_1 & \varkappa_2 \dots\dots & \varkappa_n \end{pmatrix},$$

and is seen to be true in that we can multiply both the term and $b_0{}^m \Sigma$ by $b_0{}^\lambda$.

2. The Second Formula.

By B, 1, (2),

$$\begin{pmatrix} 0^{\lambda_0} & 1^{\lambda_1} & \dots\dots & n^{\lambda_n} \\ 0^m & \varkappa_1 & \varkappa_2 \dots\dots & \varkappa_n \end{pmatrix} = (-1)^{mn} \begin{pmatrix} 0^{\lambda_n} & 1^{\lambda_{n-1}} & \dots\dots & n^{\lambda_0} \\ 0^m & (m-\varkappa_n) & (m-\varkappa_{n-1}) \dots\dots & (m-\varkappa_1) \end{pmatrix}.$$

By the first formula of derivation 1, and by B, 1, (2),

$$\begin{pmatrix} 0^{\lambda_n} & 1^{\lambda_{n-1}} & \dots\dots & n^{\lambda_0} \\ 0^m & (m-\varkappa_n) & (m-\varkappa_{n-1}) \dots\dots & (m-\varkappa_1) \end{pmatrix} = \begin{pmatrix} 0^{\lambda_n+\kappa} & 1^{\lambda_{n-1}} & \dots\dots & n^{\lambda_0} \\ 0^{m+\kappa} & (m-\varkappa_n) & (m-\varkappa_{n-1}) \dots\dots & (m-\varkappa_1) \end{pmatrix}$$

$$= (-1)^{(m+\kappa)n} \begin{pmatrix} 0^{\lambda_0} & 1^{\lambda_1} & \dots\dots & n^{\lambda_n+\kappa} \\ 0^{m+\kappa} & (\varkappa_1+\kappa) & \dots\dots & (\varkappa_n+\kappa) \end{pmatrix}.$$

Therefore by substitution,

$$\begin{pmatrix} 0^{\lambda_0} & 1^{\lambda_1} & \dots\dots & n^{\lambda_n} \\ 0_m & \varkappa_1 & \varkappa_2 \dots\dots & \varkappa_n \end{pmatrix} = (-1)^{n\kappa} \begin{pmatrix} 0^{\lambda_0} & 1^{\lambda_1} & \dots\dots & n^{\lambda_n+\kappa} \\ 0^{m+\kappa} & (\varkappa_1+\kappa) & (\varkappa_2+\kappa) \dots & (\varkappa_n+\kappa) \end{pmatrix},$$

and this is the second formula of derivation.

3. The Third Formula.

The third formula of derivation comes from the third formula of reduction in that we turn the latter about, and write n^1 instead of r, $n^1+\varkappa$ instead of n, and consequently $\varkappa$ instead of $n-r$. We then have, dropping accents,

$$\begin{pmatrix} 0^{\lambda_0} & 1^{\lambda_1} & \dots\dots & n^{\lambda_n} \\ 0^m & \varkappa_1 & \varkappa_2 \dots\dots & \varkappa_n \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0} & 1^{\lambda_1} & \dots & n^{\lambda_n} (n+1)^0 \dots & (n+\varkappa)^0 \\ 0^m & \varkappa_1 & \varkappa_2 \dots\dots & \varkappa_n & 0^\varkappa \end{pmatrix}.$$

4. The Fourth Formula.

We can obtain the fourth formula of derivation from the fourth of reduction by turning the latter about. Remembering that $\varkappa_1 = \varkappa_2 = \dots = \varkappa_r = m$, the fourth formula of reduction may be written

$$\begin{pmatrix} r^{\lambda_r} & (r+1)^{\lambda_{r+1}} & \dots\dots & n^{\lambda_n} \\ 0^m & m^r & \varkappa_{r+1} \dots\dots & \varkappa_n \end{pmatrix} = (-1)^{mr} \begin{pmatrix} 0^{\lambda_r} & 1^{\lambda_{r+1}} & \dots\dots & (n-r)^{\lambda_n} \\ 0^m & \varkappa_{r+1} & \dots\dots & \varkappa_n \end{pmatrix};$$

we may now put,